Solution 1

1. (a) It's easy to check that $(X, \|\ \|_\infty)$ is normed space. Let's prove the completeness.

Suppose $\{f_n\}$ is a Cauchy sequence in X.

Then for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$, st

$$\sup_{x \in \mathbb{R}} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty < \frac{\varepsilon}{2}, \quad \forall n, m \geq N.$$

Therefore, for each $x \in \mathbb{R}$, $\{f_m(x)\}_m$ is Cauchy in $\mathbb{R}$ and thus we can define function

$$f(x) := \lim_{m \to \infty} f_m(x), \quad x \in \mathbb{R}$$

Since N is independent of x, we can take $m \to \infty$ and Consequently, $\|f_n - f\|_\infty = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N$.

which implies $\{f_n\}$ is normly convergent to f.

Next, we prove $f \in X$.

Fix $x \in \mathbb{R}$, let $\delta > 0$ such that for any $|y - x| < \delta$,

$$|f_N(y) - f_N(x)| < \varepsilon$$

Then $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(y) - f_N(x)| + |f_N(y) - f(y)|$

$$\leq 3\varepsilon$$

Therefore, f is continuous.

Since $\|f\|_\infty \leq \|f - f_N\|_\infty + \|f_N\|_\infty < \varepsilon + \|f_N\|_\infty < \infty$

we have $f \in X$.

(b) Let $f_n(x) = \sin\left(\sqrt{x^2 + \frac{1}{n}}\right)$, $\forall n \in \mathbb{N}$.

f_n is bounded and differentiable on $\mathbb{R}$, $f_n \in Y$.

Let $f(x) = \lim_{n \to \infty} f_n(x) = \sin|x|$.

$$\|f_n - f\|_\infty = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$$
$$= \sup_{x \in \mathbb{R}} \left| \sin\left(\sqrt{x^2+\tfrac{1}{n}}\right) - \sin|x| \right|$$

Since $g(y) = \sin y$ is uniformly continuous,

For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, we have

$$\sup_{x \in \mathbb{R}} \left| \sin\sqrt{x^2+\tfrac{1}{n}} - \sin|x| \right| < \varepsilon$$

That is, $\|f_n - f\|_\infty < \varepsilon$.

However $f \notin Y$, which implies Y is not closed.

Remark: ① We must prove $\{f_n\}$ normly converges to f, but $f \notin Y$.

② We don't require $f \in X$. However, since f_n is normly convergent and X is Banach space, we have $f \in X$.

(c) The norm $\|\cdot\|_1$ and $\|\ \|_\infty$ are not equivalent on Z.

Let
$$f_n(x) = \begin{cases} 1, & x \in [-n, n] \\ -x+n+1, & x \in (n, n+1] \\ x+n+1, & x \in [-(n+1), -n) \\ 0 & x \in \mathbb{R} \setminus [-(n+1), n+1] \end{cases}, \quad n \in \mathbb{N}.$$

Then $\|f_n\|_1 = \int f_n \, dx = 2n+1 < \infty$, $n \in \mathbb{Z}$.

And $\|f_n\|_\infty = 1$, $\forall n \in \mathbb{Z}$.

We can't find a constant $c \in \mathbb{R}$, such that
$$\|f\|_1 \le c \|f\|_\infty, \quad \text{for any } f \in Z.$$